

First Passage Percolation with nonidentical passage times

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Abstract

In this paper we consider first passage percolation on the square lattice \mathbb{Z}^d with passage times that are independent and have bounded p^{th} moment for some $p > 6(1 + d)$, but not necessarily identically distributed. For integer $n \geq 1$, let $T(0, n)$ be the minimum time needed to reach the point $(n, \mathbf{0})$ from the origin. We prove that $\frac{1}{n}(T(0, n) - \mathbb{E}T(0, n))$ converges to zero in L^2 and use a subsequence argument to obtain almost sure convergence. As a corollary, for i.i.d. passage times, we also obtain the usual almost sure convergence of $\frac{T(0, n)}{n}$ to a constant μ .

Key words: First passage percolation nonidentical passage times.

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1 Introduction

Consider the square lattice \mathbb{Z}^d with edges $\{e_i\}_{i \geq 1}$. The passage times $\{t(e_i)\}_i$ are independent random variables that satisfy the following conditions.

(i) We have that $\sup_i \mathbb{P}(t(e_i) < \epsilon) \longrightarrow 0$ as $\epsilon \downarrow 0$.

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(ii) There exists a constant $\eta > 0$ such that $\sup_i \mathbb{E}(t(e_i))^{6(1+d)+\eta} < \infty$.

For $n \geq 1$, we are interested in the shortest time path from $(0, \mathbf{0})$ to $(n, \mathbf{0})$, where $\mathbf{0}$ is the $(d-1)$ -dimensional zero vector. To define such a path, we proceed as follows. For any fixed path π starting from the origin and containing k edges e_1, \dots, e_k , we define the passage time to be $T(\pi) = \sum_{i=1}^k t(e_i)$. Using (ii), we get that there exists a constant $0 < \beta_1 < \mu$ such that

$$\mathbb{P}(T(\pi) \leq \beta_1 k) \leq e^{-dk} \quad (1)$$

for all $k \geq 1$. We prove all estimates at the end of this section. By (iii) we have that $\mu = \sup_i \mathbb{E}t(e_i) < \infty$ and by (ii) we have that $\mu \geq \inf_i \mathbb{E}t(e_i) > 0$. Let E_k denote the event that there exists a path starting from $(0, \mathbf{0})$ containing $r \geq \frac{8\mu}{\beta_1}k$ edges and whose passage time is less than $\beta_1 r$. Since there are at most $(2d)^r$ paths containing r edges, we have that

$$\mathbb{P}(E_k) \leq \sum_{r \geq 8\mu\beta_1^{-1}k} (2d)^r e^{-dr} \leq C e^{-\beta_2 k} \quad (2)$$

for all $k \geq 1$ and for some positive constants β_2 and C . To obtain (2), we let $\delta = d - \log(2d)$. Since $de^{-d} \leq e^{-1} < \frac{1}{2}$ for all $d \geq 2$, we have that $\delta > 0$ and we obtain

$$\mathbb{P}(E_k) \leq \sum_{r \geq 8\mu\beta_1^{-1}k} e^{-\delta r} = \frac{1}{1 - e^{-\delta}} e^{-\delta 8\mu\beta_1^{-1}k}.$$

For $i \geq 1$, let f_i denote the edge between $(i-1, \mathbf{0})$ and $(i, \mathbf{0})$ and let $A_n = \{\sum_{i=1}^{2n} t(f_i) \leq 6\mu n\}$, where μ is as above. There exists a constant $C_1 > 0$ such that

$$\mathbb{P}(A_n^c) \leq \frac{C_1}{n^2} \quad (3)$$

for all $n \geq 1$. Finally, setting $F_n = E_n^c \cap A_n$, we note that if F_n occurs, then the time taken to reach $(i, \mathbf{0})$ from $(0, \mathbf{0})$ is less than $6\mu n$, for each $1 \leq i \leq 2n$. Since E_n^c also occurs, every path starting from $(0, \mathbf{0})$ and containing $r \geq \frac{8\mu}{\beta_1}n$ edges has passage time at least $\beta_1 r \geq 8\mu n$. Therefore, if F_n occurs, the shortest time path from $(0, \mathbf{0})$ to $(i, \mathbf{0})$ is contained in $B_{8\mu\beta_1^{-1}n} := [-8\mu\beta_1^{-1}n, 8\mu\beta_1^{-1}n]^d$ for each $1 \leq i \leq n$.

From (2) and (3), we have that

$$\mathbb{P}(F_n^c) \leq \frac{C_2}{n^2} \quad (4)$$

for some constant $C_2 > 0$ and thus by Borel-Cantelli lemma, we have that $\mathbb{P}(\liminf_n F_n) = 1$. Fix $\omega \in \liminf_n F_n$ and for every $n \geq 1$, define $T(0, n)(\omega)$ to be the shortest time taken for reaching $(n, \mathbf{0})$ from $(0, \mathbf{0})$. If there is more than one path that attains the shortest time, we provide an iterative procedure at the end of this section to choose a unique path.

We are interested in studying the convergence of $\frac{T(0, n)}{n}$. We have the following result.

Theorem 1. *We have that*

$$\frac{1}{n} (T(0, n) - \mathbb{E}T(0, n)) \longrightarrow 0 \text{ a.s. and in } L^2 \quad (5)$$

as $n \rightarrow \infty$.

For the case of independent and identically distributed (i.i.d.) random variables, we have the following Corollary.

Corollary 2. *If the passage times are i.i.d., we have that*

$$\frac{T(0, n)}{n} \longrightarrow \mu \text{ a.s. and in } L^2 \quad (6)$$

as $n \rightarrow \infty$, for some constant $\mu > 0$.

The constant μ is also called the time constant; Alexander (1993), Cox and Durrett (1981), Kesten (1993) and Smythe and Wierman (2008) and references therein contain further material on first passage percolation.

The paper is organized as follows: In the rest of this section, we prove estimates (1) and (3) and provide an iterative procedure for choosing the minimum time path. In Section 2, we prove Theorem 1 and Corollary 2.

To prove (1), we write

$$\mathbb{P}(T(\pi) \leq \beta k) = \mathbb{P}\left(\sum_{i=1}^k t(e_i) \leq \beta k\right)$$

for a fixed $\beta > 0$. Since $\{t(h_i)\}_i$ are independent, we have for a fixed $s > 0$ that

$$\mathbb{P}(T(\pi) \leq \beta k) = \mathbb{P}\left(\sum_i t(e_i) \leq \beta k\right) \leq e^{s\beta k} \prod_{i=1}^k \mathbb{E}(e^{-st(e_i)}). \quad (7)$$

For a fixed $\epsilon > 0$, we have that

$$\begin{aligned}\mathbb{E}e^{-st(e_i)} &= \int_{t(e_i) < \epsilon} e^{-st(e_i)} d\mathbb{P} + \int_{t(e_i) \geq \epsilon} e^{-st(e_i)} d\mathbb{P} \\ &\leq \int_{t(e_i) < \epsilon} e^{-st(e_i)} d\mathbb{P} + e^{-s\epsilon} \\ &\leq \mathbb{P}(t(e_i) < \epsilon) + e^{-s\epsilon}.\end{aligned}\tag{8}$$

Using (i), the first term in the last expression is less than $\frac{e^{-6d}}{2}$ if $\epsilon > 0$ is small, independent of i . Fixing such an ϵ , we choose s large so that the second term is also less than $\frac{e^{-6d}}{2}$. Substituting into (11), we have that

$$\mathbb{P}(T(\pi) \leq \beta k) \leq e^{s\beta k} e^{-3dk} \leq e^{-2dk},$$

provided $\beta > 0$ is small. We fix such a small $\beta < \mu$.

To prove (3), we let $\mu_i = \mathbb{E}t(f_i)$ and use Chebychev's inequality to write

$$\mathbb{P}(A_n^c) \leq \mathbb{P}\left(\sum_{i=1}^{2n} X_i \geq 4\mu n\right) \leq \frac{1}{(4\mu n)^4} \mathbb{E}\left(\sum_i X_i\right)^4, \tag{9}$$

where $X_i = t(f_i) - \mu_i$. Since $\{X_i\}_i$ are independent, we have that $\mathbb{E}X_i X_j = 0$ for $i \neq j$. Thus we have

$$\mathbb{E}\left(\sum_i X_i\right)^4 = \sum_i \mathbb{E}X_i^4 + \sum_{i \neq j} \mathbb{E}X_i^2 X_j^2 \leq C_1 n^2$$

for some constant $C_1 > 0$ by (ii). Substituting into (9) proves (3).

Finally, we provide an iterative procedure to choose the shortest time path in the presence of multiple choices. For simplicity we provide for $d = 2$. An analogous procedure holds for general d . Fix $\omega \in \liminf_n F_n$ and let $\mathcal{S}_1 = \{L_i\}_{1 \leq i \leq W} = \{(S_{i,1}, \dots, S_{i,H_i})\}_{1 \leq i \leq W}$ be the set of all paths with the shortest passage time from $(0, 0)$ to $(n, 0)$. We note that $W = W(\omega) < \infty$. Let $x_{i,j}$ and $y_{i,j}$ be the x - and y -coordinates, respectively, of the centre of $S_{i,j}$. Let $y'_1 = \min_{L_k \in \mathcal{S}_1} y_{k,1}$ and let $\mathcal{S}'_1 = \{L_k \in \mathcal{S}_1 : y_{k,1} = y'_1\}$. Let $x'_1 = \min_{L_k \in \mathcal{S}'_1} x_{k,1}$. Let h_1 be the edge attached to the origin whose centre has coordinates (x'_1, y'_1) . Clearly h_1 is the first edge of some path in \mathcal{S}'_1 . Let \mathcal{S}_2 be the set of paths in \mathcal{S}'_1 whose first edge is h_1 . Repeating the above procedure with \mathcal{S}_2 , we obtain an edge h_2 attached to h_1 . Continuing iteratively, this procedure terminates after a finite number of steps resulting in a unique path. Also, the final path obtained does not depend on the initial ordering of the paths.

2 Proof of Theorem 1

For $n \geq 1$, we define auxiliary random variables $\{\hat{T}_k^{(n)}\}_{k \geq 1}$ defined as follows. For $i \geq 1$, let $t_n(e_i) = \min(t(e_i), n^\alpha)$, where $\alpha < \frac{1}{6}$ is a constant to be determined later. Since $t_n(e_i) \leq t(e_i)$ a.s., we have that (i) and (ii) are satisfied by $\{t_n(e_i)\}_i$. For any fixed path π starting from the origin and containing k edges e_1, \dots, e_k , we define the passage time to be $\hat{T}_n(\pi) = \sum_{i=1}^k t_n(e_i)$. We have

$$\mathbb{P}(\hat{T}_n(\pi) \leq \beta_1 k) \leq e^{-dk} \quad (10)$$

for all $k \geq 1$. Here the constant β_1 is the same as in (1) and is independent of n . To prove (10), we use the fact that $\{t_n(e_i)\}_i$ are independent and thus for a fixed $s > 0$ we have that

$$\mathbb{P}(\hat{T}_n(\pi) \leq \beta k) = \mathbb{P}\left(\sum_i t_n(e_i) \leq \beta k\right) \leq e^{s\beta k} \prod_{i=1}^k \mathbb{E}(e^{-st_n(e_i)}). \quad (11)$$

For a fixed $0 < \epsilon < 1$, we have that

$$\begin{aligned} \mathbb{E}e^{-st_n(e_i)} &= \int_{t_n(e_i) < \epsilon} e^{-st_n(e_i)} d\mathbb{P} + \int_{t_n(e_i) \geq \epsilon} e^{-st_n(e_i)} d\mathbb{P} \\ &\leq \int_{t_n(e_i) < \epsilon} e^{-st_n(e_i)} d\mathbb{P} + e^{-s\epsilon} \\ &= \int_{t(e_i) < \epsilon} e^{-st(e_i)} d\mathbb{P} + e^{-s\epsilon} \end{aligned}$$

which is the same as (8). The final equality is because $\epsilon < 1$ and thus $t_n(e_i) < \epsilon$ if and only if $t(e_i) < \epsilon$. Following an analogous analysis following (8) we obtain (10). For $k \geq 1$, let $\hat{E}_k(n)$ denote the event that there exists a path π_1 starting from $(0, \mathbf{0})$ containing $r \geq \frac{8\mu}{\beta_1} k$ edges and whose passage time $\hat{T}_n(\pi_1)$ is less than $\beta_1 r$. As in (2) we have that

$$\mathbb{P}(\hat{E}_k(n)) \leq Ce^{-\beta_2 k} \quad (12)$$

for all $k \geq 1$, where β_2 and C are as in (2).

As before, for $i \geq 1$ let f_i denote the edge between $(i-1, \mathbf{0})$ and $(i, \mathbf{0})$ and for $k \geq 1$, let $\hat{A}_n(k) = \{\sum_{i=1}^{2n} t_k(f_i) \leq 6\mu n\}$, where μ is as above. Following an analogous analysis as in Section 1, there exists a constant $C_1 > 0$ such that

$$\mathbb{P}(\hat{A}_n^c(n)) \leq \frac{C_1}{n^2} \quad (13)$$

for all $n \geq 1$. Finally, set $\hat{F}_n = \cap_{k=1}^n \hat{E}_k^c(n) \cap \hat{A}_n(n)$ and fix $1 \leq k \leq n$. If \hat{F}_n occurs, then the time $\hat{T}_i^{(k)}$ taken to reach $(i, \mathbf{0})$ from $(0, \mathbf{0})$ is less than $6\mu n$, for each $1 \leq i \leq 2n$. This is because $t_n(f_i) \geq t_k(f_i)$ and thus $\hat{A}_n(n) \subset \hat{A}_n(k)$. Since $\hat{E}_k^c(n)$ also occurs, every path π starting from $(0, \mathbf{0})$ and containing $r \geq \frac{8\mu}{\beta_1}n$ edges has passage time $\hat{T}_k(\pi)$ at least $\beta_1 r \geq 8\mu n$. Therefore, if \hat{F}_n occurs, the shortest time path with passage time $\hat{T}_i^{(k)}$ from $(0, \mathbf{0})$ to $(i, \mathbf{0})$ is contained in $B_{8\mu\beta_1^{-1}n} := [-8\mu\beta_1^{-1}n, 8\mu\beta_1^{-1}n]^d$ for each $1 \leq i \leq 2n$ and for each $1 \leq k \leq n$.

From (12) and (13), we have that

$$\mathbb{P}(\hat{F}_n^c) \leq Cn e^{-\beta_2 n} + \frac{C_1}{n^2} \leq \frac{C_2}{n^2} \quad (14)$$

for some constant $C_2 > 0$. Thus $\mathbb{P}(\liminf_n \hat{F}_n \cap F_n) = 1$.

Fix $\omega \in \liminf_n \hat{F}_n \cap F_n$ and $m \geq 1$. For every $1 \leq k \leq 2m$, define $\hat{T}_k^{(m)} = \hat{T}_k^{(m)}(\omega)$ to be the shortest time taken for reaching $(k, 0)$ from $(0, 0)$, as in Section 1. We have the following result.

Lemma 3. *We have that*

$$\mathbb{E}(\hat{T}_n^{(n)} - \mathbb{E}\hat{T}_n^{(n)})^2 \leq C_1 n^{1+3\alpha} \quad (15)$$

for all $n \geq 1$ and some constant $C_1 > 0$.

We prove the above lemma at the end of this section. We use Lemma 3 to obtain L^2 convergence of $\frac{1}{n}(T_n - \mathbb{E}T_n)$, where $T_n = T(0, n)$.

Corollary 4.

$$\mathbb{E}(T_n - \mathbb{E}T_n)^2 \leq C_2 n^{\frac{3}{2}-\beta} \quad (16)$$

for all $n \geq 1$ and some positive constants C_2 and β .

Proof of Corollary 4: We have that

$$\mathbb{E}(T_n - \mathbb{E}T_n)^2 \leq 2I_1 + 2\mathbb{E}(\hat{T}_n^{(n)} - \mathbb{E}\hat{T}_n^{(n)})^2, \quad (17)$$

where

$$\begin{aligned} I_1 &= \mathbb{E}(T_n - \hat{T}_n^{(n)} - \mathbb{E}(T_n - \hat{T}_n^{(n)}))^2 \\ &\leq 2\mathbb{E}(T_n - \hat{T}_n^{(n)})^2 + 2(\mathbb{E}T_n - \mathbb{E}\hat{T}_n^{(n)})^2 \\ &\leq 4\mathbb{E}(T_n - \hat{T}_n^{(n)})^2. \end{aligned} \quad (18)$$

It suffices to estimate the last term.

We let G_n denote the event that the passage time $t(e_i)$ of every edge in $B_{8\mu\beta_1^{-1}n}$ is less than n^α . We have that

$$\hat{T}_n^{(n)} \mathbf{1}(H_n) = T_n \mathbf{1}(H_n). \quad (19)$$

where $H_n = G_n \cap F_n \cap \hat{F}_n$. Thus

$$\mathbb{E}(T_n - \hat{T}_n^{(n)})^2 = \mathbb{E}(T_n - \hat{T}_n^{(n)})^2 \mathbf{1}(H_n^c) \leq \left(\mathbb{E}(T_n - \hat{T}_n^{(n)})^4 \right)^{1/2} (\mathbb{P}(H_n^c))^{1/2}, \quad (20)$$

by Cauchy-Schwarz inequality. We have that

$$\mathbb{E}(T_n - \hat{T}_n^{(n)})^4 \leq 16\mathbb{E}T_n^4 + 16\mathbb{E}\left(\hat{T}_n^{(n)}\right)^4.$$

Since $T_n \leq \sum_{i=1}^n t(f_i)$, where as before, f_i denotes the edge between $(i-1, 0)$ and $(i, 0)$, we have that

$$\mathbb{E}T_n^4 \leq n^3 \sum_{i=1}^n \mathbb{E}t(f_i)^4 \leq C_1 n^4$$

for some constant $C_1 > 0$. An analogous estimate holds for $\mathbb{E}(\hat{T}_n^{(n)})^4$. Thus from (20), we have that

$$\mathbb{E}(T_n - \hat{T}_n^{(n)})^2 \leq C_2 n^2 (\mathbb{P}(H_n^c))^{1/2}, \quad (21)$$

for some constant $C_2 > 0$.

Finally, we choose $\alpha < \frac{1}{6}$ and $6(1+d) < K < 6(1+d) + \eta$ such that $K\alpha > 1+d$. Here $\eta > 0$ is as in (iii). We then have that

$$\mathbb{P}(G_n^c) \leq \sum_{i=1}^{C_3 n^d} \mathbb{P}(t(e_i) \geq n^\alpha) \leq \frac{C_3 n^d}{n^{K\alpha}} \mathbb{E}t(e_i)^K \leq \frac{C_4}{n^{1+2\delta}} \quad (22)$$

for some positive constants C_3, C_4 and δ . Thus from (4), (14) and (21), we get that

$$\mathbb{E}(T_n - \hat{T}_n^{(n)})^2 \leq C_5 n^2 n^{-\frac{1}{2}-\delta} = C_5 n^{\frac{3}{2}-\delta},$$

for some positive constant C_5 . ■

Proof of Theorem 1: We claim that it suffices to prove that $\frac{1}{n} \left(\hat{T}_n^{(n)} - \mathbb{E} \hat{T}_n^{(n)} \right)$ converges to zero a.s. Indeed, letting H_n be as in proof of Corollary 4 and using (19), we have that

$$\frac{1}{n} (T_n - \mathbb{E} T_n) = \frac{1}{n} \left(\hat{T}_n^{(n)} - \mathbb{E} \hat{T}_n^{(n)} \right) + J_{1,n} - \mathbb{E} J_{1,n} - J_{2,n} + \mathbb{E} J_{2,n},$$

where $J_{1,n} = \frac{T_n}{n} \mathbf{1}(H_n^c)$ and $J_{2,n} = \frac{\hat{T}_n^{(n)}}{n} \mathbf{1}(H_n^c)$. From (4), (22) and Borel-Cantelli Lemma we have that $\mathbb{P}(\liminf_n H_n) = 1$. Thus a.s. we have that $\limsup_n J_{1,n} = 0 = \limsup_n J_{2,n}$.

It remains to show that $\mathbb{E} J_{i,n} \rightarrow 0$ as $n \rightarrow \infty$ for $i = 1, 2$. We show that $\sup_n \mathbb{E} J_{i,n}^2 < \infty$ for $i = 1, 2$. This implies that $J_{1,n}$ and $J_{2,n}$ are uniformly integrable and completes the claim. We have that

$$J_{1,n} \leq \frac{T_n}{n} \leq \frac{1}{n} \sum_{i=1}^n t(f_i)$$

where as before f_i denotes the edge from $(i-1, 0)$ to $(i, 0)$. Thus

$$\mathbb{E} J_{1,n}^2 \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} t(f_i)^2 \leq C_1$$

for some constant $C_1 > 0$ by condition (iii). An analogous estimate holds for $J_{2,n}$.

To prove that $\frac{1}{n} \left(\hat{T}_n^{(n)} - \mathbb{E} \hat{T}_n^{(n)} \right)$ converges to zero a.s., we use a subsequence argument as follows. Set $S_n = \hat{T}_n^{(n)} - \mathbb{E} \hat{T}_n^{(n)}$. From Lemma 3, we have that $\mathbb{E} S_n^2 \leq C_1 n^{1+3\alpha}$. Thus for a fixed $\epsilon > 0$, we have that

$$\mathbb{P}(|S_{n^2}| > n^2 \epsilon) \leq \frac{\mathbb{E} S_{n^2}^2}{\epsilon^2 n^4} \leq \frac{C_2}{n^{2-6\alpha}}$$

for some constant $C_2 > 0$. Since $\alpha < \frac{1}{6}$, we have that $2 - 6\alpha > 1$ and by Borel-Cantelli Lemma, we have that $\frac{S_{n^2}}{n^2} \rightarrow 0$ a.s. as $n \rightarrow \infty$.

We now set $D_{n^2} = \max_{n^2 \leq k < (n+1)^2} |S_k - S_{n^2}|$ and estimate D_{n^2} as follows. For $n^2 \leq k < (n+1)^2$, we write

$$\begin{aligned} |S_k - S_{n^2}| &\leq |\hat{T}_k^{(k)} - \hat{T}_{n^2}^{(n^2)}| + \mathbb{E} |\hat{T}_k^{(k)} - \hat{T}_{n^2}^{(n^2)}| \\ &\leq |\hat{T}_k^{(k)} - \hat{T}_{n^2}^{(k)}| + |\hat{T}_{n^2}^{(k)} - \hat{T}_{n^2}^{(n^2)}| \\ &\quad + \mathbb{E} |\hat{T}_k^{(k)} - \hat{T}_{n^2}^{(k)}| + \mathbb{E} |\hat{T}_{n^2}^{(k)} - \hat{T}_{n^2}^{(n^2)}|. \end{aligned} \quad (23)$$

For any integers $k_1 < k_2 < k_3$, we have that

$$\hat{T}_{k_1, k_3}^{(k)} \leq \hat{T}_{k_1, k_2}^{(k)} + \hat{T}_{k_2, k_3}^{(k)} \text{ and } \hat{T}_{k_1, k_2}^{(k)} \leq \hat{T}_{k_1, k_3}^{(k)} + \hat{T}_{k_2, k_3}^{(k)}. \quad (24)$$

Here $\hat{T}_{k_1, k_2}^{(k)}$ denotes minimum passage time to go from $(k_1, 0)$ to $(k_2, 0)$ and is defined analogously as $\hat{T}_n^{(k)}$ for each k_1 and k_2 . Thus

$$|\hat{T}_k^{(k)} - \hat{T}_{n^2}^{(k)}| \leq \hat{T}_{k, n^2}^{(k)} \leq k^\alpha (k - n^2) \leq (n+1)^{2\alpha} ((n+1)^2 - n^2) \leq C_1 n^{1+2\alpha}$$

for some constant $C_1 > 0$. The second inequality is true since the passage time of every edge is less than k^α . Substituting the above estimate into (23), we obtain that

$$|S_k - S_{n^2}| \leq 2C_1 n^{1+2\alpha} + |\hat{T}_{n^2}^{(k)} - \hat{T}_{n^2}^{(n^2)}| + \mathbb{E}|\hat{T}_{n^2}^{(k)} - \hat{T}_{n^2}^{(n^2)}|. \quad (25)$$

To estimate the remaining terms, we note that

$$0 \leq \hat{T}_{n^2}^{(k)} - \hat{T}_{n^2}^{(n^2)} \leq \hat{T}_{n^2}^{((n+1)^2)} - \hat{T}_{n^2}^{(n^2)} =: I_{n^2}$$

since $n^2 \leq k < (n+1)^2$. Thus

$$\frac{D_{n^2}}{n^2} \leq \frac{2C_1}{n^{1-2\alpha}} + \frac{I_{n^2}}{n^2} + \frac{\mathbb{E}I_{n^2}}{n^2}.$$

We claim that $\frac{I_{n^2}}{n^2} \rightarrow 0$ a.s. and that $\frac{I_{n^2}}{n^2}$ is uniformly integrable. Assuming the claims for the moment, we get that $\frac{D_{n^2}}{n^2} \rightarrow 0$ a.s. as $n \rightarrow \infty$. For $n^2 \leq k < (n+1)^2$, we have that

$$\frac{|S_k|}{k} \leq \frac{|S_k - S_{n^2}|}{k} + \frac{|S_{n^2}|}{k} \leq \frac{|S_k - S_{n^2}|}{n^2} + \frac{|S_{n^2}|}{n^2} \leq \frac{D_{n^2}}{n^2} + \frac{|S_{n^2}|}{n^2}.$$

This proves that the original sequence $\frac{S_k}{k} \rightarrow 0$ a.s. as $k \rightarrow \infty$.

To prove the two claims regarding I_{n^2} , we note that

$$\hat{T}_{n^2}^{((n+1)^2)} \mathbf{1}(\hat{H}_n) = \hat{T}_{n^2}^{(n^2)} \mathbf{1}(\hat{H}_n)$$

where $\hat{H}_n = \hat{F}_{n^2} \cap \hat{F}_{(n+1)^2} \cap \hat{G}_{n^2}$ and \hat{G}_{n^2} is the event that the passage time $t(e_i)$ of every edge in $B_{20\mu\beta_1^{-1}n^2}$ is less than $n^{2\alpha}$. As in (22) we have that $\mathbb{P}(\hat{G}_{n^2}^c) \leq \frac{C_1}{n^{2+\delta_2}}$ for some constant $\delta_2 > 0$. From (14) and Borel-Cantelli

lemma, we then have that $\mathbb{P}(\liminf_n \hat{H}_n) = 1$. Since $I_{n^2} = I_{n^2} \mathbf{1}(\hat{H}_n^c)$, we get that $\frac{I_{n^2}}{n^2} \rightarrow 0$ a.s. as $n \rightarrow \infty$.

To prove the uniform integrability of $\frac{I_{n^2}}{n^2}$, we note that

$$0 \leq \frac{I_{n^2}}{n^2} \leq \frac{\hat{T}^{((n+1)^2)}}{n^2} \leq \frac{1}{n^2} \sum_{i=1}^{n^2} t(f_i) =: M_n$$

where as before f_i denotes the edge from $(i-1, 0)$ to $(i, 0)$. Since $\mathbb{E}M_n^2 \leq \frac{1}{n^2} \sum_{i=1}^{n^2} \mathbb{E}t(f_i)^2 \leq C_1$ for some constant $C_1 > 0$, we are done. \blacksquare

Proof of Corollary 2: We show that $\frac{\mathbb{E}T(0, n)}{n} \rightarrow \mu$ for some constant $\mu > 0$. Since

$$\mathbb{E}T(0, n+m) \leq \mathbb{E}T(0, n) + \mathbb{E}T(m, m+n) = \mathbb{E}T(0, n) + \mathbb{E}T(0, m),$$

we have by Fekete's Lemma that

$$\lim_n \frac{\mathbb{E}T(0, n)}{n} = \inf_{n \geq 1} \frac{\mathbb{E}T(0, n)}{n} =: \mu.$$

To show that $\mu > 0$, we note that if A_{0, k_1}^c occurs for $k_1 = \beta_1(8\mu)^{-1}n$, then every path containing $r \geq 8\mu\beta_1^{-1}k_1 \geq n$ edges has passage time at least $\beta_1 r \geq 8\mu k_1 \geq \beta_1 n$. Thus

$$\mathbb{E}T(0, n) \geq \beta_1 n \mathbb{P}(A_{0, k_1}^c) \geq \beta_3 n$$

for all $n \geq 1$ and some constant $\beta_3 > 0$, by (2). \blacksquare

Proof of Lemma 3: We order the edges as e_1, e_2, \dots and for each $i \geq 1$, set $\mathcal{F}_i = \sigma(\hat{t}(e_l) : 1 \leq l \leq i)$. For $l \geq 1$, let $X_l = \mathbb{E}(\hat{T}_n^{(n)} | \mathcal{F}_l) - \mathbb{E}(\hat{T}_n^{(n)} | \mathcal{F}_{l-1})$. We have that $0 \leq \hat{T}_n^{(n)} \leq \sum_{i=1}^n t_n(f_i) \leq n^{1+\alpha}$ a.s., where as before f_i denotes the edge from $(i-1, \mathbf{0})$ to $(i, \mathbf{0})$. Thus we have by Levy's martingale convergence theorem that

$$Y_m := \sum_{l=1}^m X_l = \mathbb{E}(\hat{T}_n^{(n)} | \mathcal{F}_m) - \mathbb{E}\hat{T}_n^{(n)} \longrightarrow \hat{T}_n^{(n)} - \mathbb{E}(\hat{T}_n^{(n)}) \text{ a.s.}$$

as $m \rightarrow \infty$. By Dominated convergence theorem, we then have that

$$\mathbb{E}(\hat{T}_n^{(n)} - \mathbb{E}\hat{T}_n^{(n)})^2 = \mathbb{E} \left(\lim_m Y_m \right)^2 = \lim_m \mathbb{E}Y_m^2.$$

By the martingale property, we have that $\mathbb{E}Y_m^2 = \sum_{l=1}^m \mathbb{E}X_l^2$. We claim that

$$X_l^2 \leq 2n^{2\alpha} (\mathbb{P}(e_l \in \pi_n | \mathcal{F}_l) + \mathbb{P}(e_l \in \pi_n | \mathcal{F}_{l-1})) \quad a.s. \quad (26)$$

where π_n is the shortest time path from $(0, 0)$ to $(n, 0)$. We prove the above result at the end. Using (26), we obtain that

$$\begin{aligned} \mathbb{E}(\hat{T}_n^{(n)} - \mathbb{E}\hat{T}_n^{(n)})^2 &\leq 2n^{2\alpha} \sum_{l=1}^{\infty} \mathbb{E} (\mathbb{P}(e_l \in \pi_n | \mathcal{F}_l) + \mathbb{P}(e_l \in \pi_n | \mathcal{F}_{l-1})) \\ &= 4n^{2\alpha} \sum_{l=1}^{\infty} \mathbb{P}(e_l \in \pi_n) \\ &= 4n^{2\alpha} \mathbb{E} \sum_{l=1}^{\infty} \mathbf{1}(e_l \in \pi_n) \\ &= 4n^{2\alpha} \mathbb{E}(\#\pi_n), \end{aligned}$$

where $\mathbf{1}(\cdot)$ refers to the indicator function.

To estimate the length of π_n , let $\mu = \sup_i \mathbb{E}t(e_i)$ be as in Section 1. We note that if $\hat{E}_k^c(n)$ occurs (see paragraph prior to (12)) for $k \geq \mu^{-1}n^{1+\alpha}$, then every path π with length $r \geq \frac{8\mu}{\beta_1}k$ has passage time $\hat{T}_n(\pi)$ at least $\beta_1 r \geq 8n^{1+\alpha}$. Since π_n has passage time at most $n^{1+\alpha}$, we obtain for $k \geq \mu^{-1}n^{1+\alpha}$ that

$$\mathbb{P}(\#\pi_n \geq 8\mu\beta_1^{-1}k) \leq \mathbb{P}(\hat{E}_k(n)) \leq e^{-\beta_2 k},$$

where $\beta_2 > 0$ is as in (12). Since $\mathbb{E}(\#\pi_n) \leq \sum_{k \geq 1} \mathbb{P}(\#\pi_n \geq k)$, we obtain that $\mathbb{E}(\#\pi_n) \leq C_1 n^{1+\alpha}$ for some constant $C_1 > 0$.

To estimate X_l , we use the notation of Kesten (1993); for $j \geq 1$, let $\nu_j(\cdot)$ denote the probability measure associated with $(\hat{t}(e_j), \hat{t}(e_{j+1}), \dots)$. Let $(\sigma_1, \sigma_2, \dots)$ and $(\omega_1, \omega_2, \dots)$ be independent realizations of $(\hat{t}(e_1), \hat{t}(e_2), \dots)$ and for $l \geq 1$, define $[\omega, \sigma]_l = (\omega_1, \omega_2, \dots, \omega_l, \sigma_{l+1}, \sigma_{l+2}, \dots)$. We have that

$$X_l = \int \nu_l(d\sigma) (T_n([\omega, \sigma]_l) - T_n([\omega, \sigma]_{l-1})).$$

We note that changing the passage time of edge e_l does not change the value of the minimum passage time by more than n^α . Also, a change occurs only if $e_l \in \pi_n([\omega, \sigma]_l)$ or $e_l \in \pi_n([\omega, \sigma]_{l-1})$. Moreover, if Thus

$$|\hat{T}_n^{(n)}([\omega, \sigma]_l) - \hat{T}_n^{(n)}([\omega, \sigma]_{l-1})| \leq n^\alpha (\mathbf{1}(e_l \in \pi_n([\omega, \sigma]_l)) + \mathbf{1}(e_l \in \pi_n([\omega, \sigma]_{l-1})))$$

and by Cauchy-Schwarz inequality, we have a.s. that

$$\begin{aligned}
X_l^2 &\leq \int \nu_l(d\sigma) |\hat{T}_n^{(n)}([\omega, \sigma]_l) - \hat{T}_n^{(n)}([\omega, \sigma]_{l-1})|^2 \\
&\leq 2n^{2\alpha} \int \nu_l(d\sigma) (\mathbf{1}(e_l \in \pi_n([\omega, \sigma]_l)) + \mathbf{1}(e_l \in \pi_n([\omega, \sigma]_{l-1}))) \\
&= 2n^{2\alpha} (\mathbb{P}(e_l \in \pi_n|\mathcal{F}_l) + \mathbb{P}(e_l \in \pi_n|\mathcal{F}_{l-1})).
\end{aligned}$$

This proves (26). ■

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